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Front-Form Hamiltonian and BRST Formulations of the Schwinger Model

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The Hamiltonian and BRST formulations of the Schwinger model are investigated in the light-front frame.

1. INTRODUCTION

In a recent paper[1] we studied the Hamiltonian[2] and Becchi-Rouet-Stora and Tyutin (BRST)[3,4] formulations of the electrodynamics in one-space, one-time dimension with massless fermions, known as the Schwinger model,[1,5–8] using the instant form of dynamics.[1,9] In the present paper we investigate the front form of dynamics[9] for the Hamiltonian[2] and BRST[3,4] formulations of this model.[1] The Schwinger model is characterized by its exact solvability, a property which is ensured by a remarkable feature of one-dimensional fermion systems, namely, that they can be described in terms of canonical one-dimensional boson fields.[3] This fermion–boson equivalence has led to the discovery of many interesting features of two-dimensional field theories.[5–8]

The Schwinger model in the instant form[9] of dynamics is seen[1] to describe a gauge-invariant theory possessing a set of *two* first-class constraints, involving one primary and one secondary constraint (cf. ref. 1 for details). In the front form[9] the model is again seen to describe a gauge-invariant theory, now possessing, however, a set of *three* first-class constraints, involving two primary constraints and one secondary constraint. This is in

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contrast with the instant form of dynamics for the theory,[1,9] where the model is seen to possess a set of two first-class constraints.[1]

In the present work we study the Hamiltonian[2] and BRST formulations[3,4] of the Schwinger model in the light-front frame with some specific gauge choices (Sections 2 and 3).

Further, in the usual Hamiltonian formulation of a gauge-invariant theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under the gauge fixing). To achieve the quantization of a gauge-invariant theory such that the gauge invariance of the theory is maintained even under gauge fixing, one goes to a more generalized procedure called the BRST formulation.[3,4] In the BRST formulation[3,4] of a gauge-invariant theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the gauge-invariant theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators having different statistics. In view of this, one introduces new anticommuting variables c and $\bar{c}$ called the Faddeev–Popov ghost and antighost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field.[3,4] In the BRST formulation, one thus embeds a gauge-invariant theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as with the anti-BRST charge operator $\bar{Q}$; the new symmetry of the quantum system (the BRST symmetry) that replaces the gauge invariance is maintained (even under the gauge fixing) and hence, projecting any state onto the sector of BRST- and anti-BRST-invariant states yields a theory which is isomorphic to the original gauge-invariant theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian are guaranteed by the conservation and nilpotency of the BRST charge Q. The Hamiltonian formulation of the theory is considered in Section 2, and its BRST formulation is studied in Section 3.

2. THE FRONT-FORM HAMILTONIAN FORMULATION

The Schwinger model in one-space, one-time dimension is described by the Lagrangian density[1,5]

$$\tilde{\mathcal{L}} := \bar{\psi}\gamma^{\mu}(i\partial_{\mu} + gA_{\mu})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}; \qquad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \quad (2.1)$$

which is equivalent to its bosonized form[1,7]:

$$\tilde{\mathcal{L}} := \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - g\varepsilon^{\mu\nu}\partial_\mu\phi A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.2a}$$

$$g^{\mu\nu} := \mathrm{diag}(+1, -1); \quad \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}; \quad \varepsilon^{01} = +1; \quad \mu,\nu = 0, 1 \tag{2.2b}$$

In component form (2.2) can be written as

$$\tilde{\mathcal{L}} = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) + g(\phi' A_0 - \dot{\phi} A_1) + \frac{1}{2}(\dot{A}_1 - A_0')^2 \tag{2.3}$$

where overdots and primes denote time and space derivatives, respectively. Equation (2.3) describes the theory in the instant form[1] and is seen to possess a set of two first-class constraints[1,2]:

$$\Omega_1 = \Pi_0 \approx 0 \qquad \text{and} \qquad \Omega_2 = (E' + g\phi') \approx 0 \tag{2.4}$$

where Ω_1 is a primary constraint and Ω_2 is a secondary constraint. The Hamiltonian and BRST formulations of this theory in the instant form have been studied in ref. 1. In the light-front frame approach one defines the coordinates [9]

$$x^{\pm} := \frac{1}{\sqrt{2}}(x^0 \pm x^1)$$

and then writes all the quantities involved in the Lagrangian density in terms of $x^{\pm}$ instead of x^0 and x^1. After doing this the Lagrangian density $\tilde{\mathcal{L}}$ in the light-front frame reads[9,1,5]

$$\mathcal{L} = (\partial_+\phi)(\partial_-\phi) + g(\partial_+\phi)A^+ - g(\partial_-\phi)A^- + \frac{1}{2}(\partial_+A^+ - \partial_-A^-)^2 \tag{2.5}$$

where

$$A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_1) \qquad \text{and} \qquad \partial_\pm\phi = \frac{1}{\sqrt{2}}(\dot{\phi} \pm \phi') \tag{2.6}$$

In the following we consider the Hamiltonian formulation of the theory described by the Lagrangian density $\mathcal{L}$, (2.5). The Euler–Lagrange equations obtained from the $\mathcal{L}$ (2.5) are

$$g(\partial_+A^+ - \partial_-A^-) = -2\partial_+\partial_-\phi \tag{2.7a}$$

$$gJ_+ = \partial_+(\partial_+A^+ - \partial_-A^-) = g\partial_+\phi \tag{2.7b}$$

$$gJ_- = -\partial_-(\partial_+A^+ - \partial_-A^-) = -g\partial_-\phi \tag{2.7c}$$

The vector current (J_μ) is seen to be conserved, i.e.,

$$\partial_\nu J^\nu := \partial_+ J_- + \partial_- J_+$$

$$= \partial_- \partial_+ \phi - \partial_+ \partial_- \phi = 0 \tag{2.8a}$$

implying that the theory possesses (at the classical level) a vector-gauge symmetry. The divergence of the axial-vector current J_5^μ at the same time is nonzero:

$$\partial_\mu J_5^\mu := \frac{1}{2} \varepsilon^{\mu\nu} F_{\mu\nu} = -(\partial_+ A^+ - \partial_- A^-) = \frac{2}{g} \partial_+ \partial_- \phi \neq 0 \tag{2.8b}$$

The nonzero divergence of the axial-vector current expressed by the axial-anomaly equation (2.8b) signifies the *absence* of the axial-vector gauge symmetry in the theory.

The light-cone canonical momenta obtained from the $\mathscr{L}$ of (2.5) are

$$\Pi^+ = \frac{\partial \mathscr{L}}{\partial(\partial_+ A^-)} = 0 \tag{2.9a}$$

$$\Pi^- = \frac{\partial \mathscr{L}}{\partial(\partial_+ A^+)} = (\partial_+ A^+ - \partial_- A^-) \tag{2.9b}$$

$$\Pi = \frac{\partial \mathscr{L}}{\partial(\partial_+ \phi)} = \partial_- \phi + g A^+ \tag{2.9c}$$

Here Π^+, Π^-, and Π are the momenta canonically conjugate respectively to A^-, A^+, and ϕ. Equations (2.9) imply that the theory possesses two primary constraints:

$$\chi_1 = (\Pi^+) \approx 0 \tag{2.10a}$$

$$\chi_2 = (\Pi - \partial_- \phi - g A^+) \approx 0 \tag{2.10b}$$

The canonical Hamiltonian density corresponding to the $\mathscr{L}$ of (2.5) is

$$\mathscr{H}_c := \Pi^+ (\partial_+ A^-) + \Pi^- (\partial_+ A^+) + \Pi(\partial_+ \phi) - \mathscr{L}$$

$$= \frac{1}{2} (\Pi^-)^2 + \Pi^- (\partial_- A^-) + g(\partial_- \phi) A^- \tag{2.11}$$

After including the primary constraints χ_1 and χ_2 in the canonical Hamiltonian density $\mathscr{H}_c$ of (2.11) with the help of Lagrange multipliers u and ϑ, we can write the total Hamiltonian density $\mathscr{H}_T$ as[2]

$$\mathscr{H}_T = \frac{1}{2} (\Pi^-)^2 + \Pi^- (\partial_- A^-) + g(\partial_- \phi) A^- + \Pi^+ u$$

$$+ (\Pi - \partial_- \phi - g A^+) \vartheta \tag{2.12}$$

The Hamilton equations obtained from the total Hamiltonian $H_T = \int \mathcal{H}_T \, dx^-$ are

$$\partial_+\phi = \frac{\partial H_T}{\partial \Pi} = v; \qquad -\partial_+\Pi = \frac{\partial H_T}{\partial \phi} = -g\partial_- A^- \pm \partial_- v \tag{2.13a}$$

$$\partial_+ A^- = \frac{\partial H_T}{\partial \Pi^+} = u; \qquad -\partial_+\Pi^- = \frac{\partial H_T}{\partial A^+} = -gv \tag{2.13b}$$

$$\partial_+ A^+ = \frac{\partial H_T}{\partial \Pi^-} = \Pi^- + \partial_- A^-; \qquad -\partial_+\Pi^+ = \frac{\partial H_T}{\partial A^-} = -\partial_-\Pi^- + g\partial_-\phi \tag{2.13c}$$

$$\partial_+ u = \frac{\partial H_T}{\partial \Pi_u} = 0; \qquad -\partial_+\Pi_u = \frac{\partial H_T}{\partial u} = \Pi \tag{2.13d}$$

$$\partial_+\vartheta = \frac{\partial H_T}{\partial \Pi_\vartheta} = 0; \qquad -\partial_+\Pi_\vartheta = \frac{\partial H_T}{\partial \vartheta} = \Pi - \partial_-\phi - gA^+ \tag{2.13e}$$

These are the equations of motion of the theory that preserve the constraints of the theory in the course of time. For the light-cone equal-time ($x^+ = y^+$) Poisson bracket $\{.,.\}_P$ of two functions A and B, we choose the convention

$$\{A(x), B(y)\}_P = \int dz^- \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \qquad 2.14)$$

Demanding that the primary constraint χ_1 be preserved in the course of time, one obtains the secondary constraint

$$\chi_3 = \{\chi_1, \mathcal{H}_T\}_P = (\partial_-\Pi^- - g(\partial_-\phi)) \approx 0 \tag{2.15}$$

The preservation of χ_2 and χ_3 for all time does not give rise to any further constraints. The theory is thus seen to possess only three constraints χ_1, χ_2, and χ_3. The matrix of the Poisson brackets of the constraints χ_i is

$$S_{\alpha\beta}(w, z) = \{\chi_\alpha(w), \chi_\beta(z)\}_P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\partial_-\delta(w^- - z^-) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.16}$$

The matrix $S_{\alpha\beta}$ is clearly singular, implying that the set of constraints χ_i is first-class and that the theory described by the $\mathcal{L}$ of (2.5) is a gauge-invariant theory. The Lagrangian density $\mathcal{L}$ of (2.5) is in fact seen to be invariant under the time-dependent gauge transformations:

$$\delta A^+ = \partial_-\beta, \quad \delta A^- = \partial_+\beta, \quad \delta\phi = 0, \quad \delta u = \partial_+\partial_+\beta, \quad \delta\vartheta = 0 \tag{2.17}$$

$$\delta\Pi^+ = 0, \quad \delta\Pi^- = 0, \quad \delta\Pi = g\partial_-\beta; \qquad \delta\Pi_u = 0, \quad \delta\Pi_\vartheta = 0 \tag{2.18}$$

up to a total divergence

$$\delta\mathcal{L} = -\, g\varepsilon^{\mu\nu}\, \partial_\mu(\beta\partial_\nu\phi) \tag{2.19}$$

where $\beta = \beta(x^-, x^+)$ is an arbitrary function of the coordinates. The action $S = \int \mathcal{L}\, dx^-\, dx^+$ is therefore gauge invariant. The reduced Hamiltonian of the theory $H_R = \int \mathcal{H}_R\, dx^-$, obtained from $H_T = \int \mathcal{H}_T\, dx^-$ after the implementation of constraints χ_i, is given by[2]

$$H_R = \int \mathcal{H}_R\, dx^- = \int dx^- \left[\frac{1}{2}(\Pi^-)^2\right] \tag{2.20}$$

H_R is thus seen to be positive semidefinite. In order to quantize the theory using Dirac's procedure,[1,2] we convert the set of first-class constraints of the theory χ_i into a set of second-class constraints by imposing arbitrarily some additional constraints on the system called gauge-fixing conditions or gauge constraints. For this purpose, for the present theory we can choose, for example, the following set of gauge-fixing conditions: (A) $A^- = 0$ and $A^+ = 0$, and (B) $A^- = 0$ and $\partial_- A^+ = 0$. Corresponding to these choices of the gauge-fixing conditions, we have the following two sets of constraints under which the quantization of the theory can be studied:

For set (A)

$$\xi_1 = \chi_1 = (\Pi^+) \approx 0 \tag{2.21a}$$

$$\xi_2 = \chi_2 = (\Pi - \partial_-\phi - gA^+) \approx 0 \tag{2.21b}$$

$$\xi_3 = \chi_3 = (\partial_-\Pi^- - g\partial_-\phi) \approx 0 \tag{2.21c}$$

$$\xi_4 = (A^-) \approx 0 \tag{2.21d}$$

$$\xi_5 = (A^+) \approx 0 \tag{2.21e}$$

and for set (B)

$$\eta_1 = \chi_1 = (\Pi^+) \approx 0 \tag{2.22a}$$

$$\eta_2 = \chi_2 = (\Pi - \partial_-\phi - gA^+) \approx 0 \tag{2.22b}$$

$$\eta_3 = \chi_3 = (\partial_-\Pi^- - g\partial_-\phi) \approx 0 \tag{2.22c}$$

$$\eta_4 = (A^-) \approx 0 \tag{2.22d}$$

$$\eta_5 = (\partial_- A^+) \approx 0 \tag{2.22e}$$

We now calculate the Poisson brackets among the set of constraints ξ_i and η_i and obtain the matrices

$$A_{\alpha\beta}(w,z) = \{\xi_\alpha(w), \xi_\beta(z)\}_P$$
$$= \begin{bmatrix} 0 & 0 & 0 & -\delta(w^- - z^-) & 0 \\ 0 & -2\partial_-\delta(w^- - z^-) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_-\delta(w^- - z^-) \\ \delta(w^- - z^-) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_-\delta(w^- - z^-) & 0 & 0 \end{bmatrix} \tag{2.23}$$

and

$$B_{\alpha\beta}(w,z) = \{\eta_\alpha(w), \eta_\beta(z)\}_P$$
$$= \begin{bmatrix} 0 & 0 & 0 & -\delta(w^- - z^-) & 0 \\ 0 & -2\partial_-\delta(w^- - z^-) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_-\partial_-\delta(w^- - z^-) \\ \delta(w^- - z^-) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_-\partial_-\delta(w^- - z^-) & 0 & 0 \end{bmatrix} \tag{2.24}$$

with the inverses

$$A^{-1}_{\alpha\beta}(w,z)$$
$$= \begin{bmatrix} 0 & 0 & 0 & \delta(w^- - z^-) & 0 \\ 0 & -\frac{1}{4}\varepsilon(w^- - z^-) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\varepsilon(w^- - z^-) \\ -\delta(w^- - z^-) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\varepsilon(w^- - z^-) & 0 & 0 \end{bmatrix} \tag{2.25}$$

and

$$B^{-1}_{\alpha\beta}(w,z)$$
$$= \begin{bmatrix} 0 & 0 & 0 & \delta(w^- - z^-) & 0 \\ 0 & -\frac{1}{4}\varepsilon(w^- - z^-) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}|w^- - z^-| \\ -\delta(w^- - z^-) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}|w^- - z^-| & 0 & 0 \end{bmatrix} \tag{2.26}$$

with

$$\int dz^- A(x,z)A^{-1}(z,y) = \int dz^- B(x,z)B^{-1}(z,y) = 1_{5\times 5}\delta(x^- - y^-)$$

Here $\varepsilon(w^- - z^-)$ is a step function defined as

$$\varepsilon(w^- - z^-) = \begin{cases} +1, & (w^- - z^-) > 0 \\ -1, & (w^- - z^-) < 0 \end{cases} \tag{2.28}$$

The Dirac bracket $\{.,.\}_D$ of the two functions A and B is defined as[2]

$$\{A, B\}_D = \{A, B\}_P - \int\int dw^- dz^-$$
$$\times \sum_{\alpha,\beta} [\{A, \Gamma_\alpha(w)\}_P [\Delta^{-1}_{\alpha\beta}(w, z)] \{\Gamma_\beta(z), B\}_P] \tag{2.29}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(w, z)$ [$= \{\Gamma_\alpha(w), \Gamma_\beta(z)\}_P$] is the matrix of the Poisson brackets of the constraints Γ_i. The transition to quantum theory is made by replacing the Dirac brackets by the operator commutation relations according to

$$\{A, B\}_D \to (-i)\,[A, B], \qquad i = \sqrt{-1} \tag{2.30}$$

Finally, the nonvanishing equal-light-cone-time ($x^+ = y^+$) commutators of the theory in case (A) i.e., in the gauge $A^- = 0$ and $A^+ = 0$, are obtained as

$$[\phi(x), \Pi(y)] = \frac{3}{2} i\delta(x^- - y^-) \tag{2.31a}$$

$$[\Pi(x), \Pi^-(y)] = \frac{1}{2} gi\delta(x^- - y^-) \tag{2.31b}$$

$$[A^+(x), \Pi^-(y)] = 2i\delta(x^- - y^-) \tag{2.31c}$$

$$[\phi(x), \Pi^-(y)] = -\frac{1}{4} gi\varepsilon(x^- - y^-) \tag{2.31d}$$

$$[\phi(x), \phi(y)] = -\frac{1}{4} i\varepsilon\,(x^- - y^-) \tag{2.31e}$$

$$[\Pi^-(x), \Pi^-(y)] = -\frac{1}{4} g^2 i\varepsilon(x^- - y^-) \tag{2.31f}$$

$$[\Pi(x), \Pi(y)] = -\frac{1}{2} \partial_-\delta\,(x^- - y^-) \tag{2.31g}$$

The nonvanishing equal-light-cone-time commutators of the theory in case (B), i.e., in the gauge $A^- = 0$ and $\partial_- A^+ = 0$ are seen to be identical with those of case (A), as they should be, and are also given by (2.31) (cf. Appendix).[1] This is not surprising (as also explained in the Appendix) in the context of the present theory considered in the instant form of dynamics[1] in view of the fact that the gauges $A^+ = 0$ and $\partial_- A^+ = 0$ conceptually mean the same.[10]

For later use, for considering the BRST formulaton of the theory described by $\mathcal{L}$, we convert the total Hamiltonian density $\mathcal{H}_T$ into the first-order Lagrangian density $\mathcal{L}_{\text{IO}}$:

$$\begin{aligned}\mathcal{L}_{\text{IO}} &= \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi(\partial_+\phi) + \Pi_u(\partial_+ u) + \Pi_\vartheta(\partial_+\vartheta) - \mathcal{H}_T \\ &= \Pi^-(\partial_+ A^+) + \Pi_u(\partial_+ u) + \Pi_\vartheta(\partial_+\vartheta) - \frac{1}{2}(\Pi^-)^2 - \Pi^-(\partial_- A^-) \\ &\quad -gA^-(\partial_-\phi) + (\partial_-\phi + gA^+)(\partial_+\phi) \end{aligned} \tag{2.32}$$

In (2.32), the terms $\Pi^+(\partial_+ A^- - u)$ and $\Pi(\partial_+\phi - \vartheta)$ drop out in view of the Hamilton equations (2.13b) and (2.13a).

3. THE BRST FORMULATION

3.1. The BRST Invariance

For the BRST formulation of the Schwinger model, we rewrite the theory as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of the gauge-invariant Schwinger model and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics; we then introduce new anticommuting variable c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b such that

$$\hat{\delta}\phi = 0, \quad \hat{\delta}A^+ = \partial_- c, \quad \hat{\delta}A^- = \partial_+ c, \quad \hat{\delta}u = \partial_+\partial_+ c, \quad \hat{\delta}\vartheta = 0 \tag{3.1a}$$

$$\hat{\delta}\Pi = g(\partial_- c), \quad \hat{\delta}\Pi^+ = 0, \quad \hat{\delta}\Pi^- = 0, \quad \hat{\delta}\Pi_u = 0, \quad \hat{\delta}\Pi_\vartheta = 0 \tag{3.1b}$$

$$\hat{\delta}c = 0, \quad \hat{\delta}\bar{c} = b\ \hat{\delta}b = 0 \tag{3.1c}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function

$$f(\phi, A^+, A^-, u, \vartheta, c, \bar{c}, b, \Pi, \Pi^+, \Pi^-, \Pi_u, \Pi_\vartheta, \Pi_c, \Pi_{\bar{c}}, \Pi_b)$$

such that $\hat{\delta}f = 0$.

3.2. Gauge-Fixing in the BRST Formalism

Performing gauge-fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{IO}$ a trivial BRST-invariant function.[3] We thus write the quantum Lagrangian density (taking, e.g., a trivial BRST-invariant function) as follows[4]:

$$\begin{aligned}\mathscr{L}_{BRST} &= \mathscr{L}_{IO} + \hat{\delta}\left[\bar{c}\left(\partial_+ A^+ + \frac{1}{2}b - gA^+ + \Pi\right)\right] \\ &= \Pi^-(\partial_+ A^+) + \Pi_u(\partial_+ u) + \Pi_\vartheta(\partial_+\vartheta) - \frac{1}{2}(\Pi^-)^2 \\ &\quad - \Pi^-(\partial_- A^-) - gA^-(\partial_-\phi) + (\partial_-\phi + gA^+)(\partial_+\phi) \\ &\quad + \hat{\delta}\left[\bar{c}\left(\partial_+ A^- + \frac{1}{2}b - gA^+ + \Pi\right)\right] \end{aligned} \tag{3.2}$$

The last term in equation (3.2) is the extra BRST-invariant gauge-fixing term. Using the definition of $\hat{\delta}$, we can rewrite $\mathscr{L}_{BRST}$ (with one integration by parts):

$$\begin{aligned}\mathscr{L}_{BRST} &= \Pi^-(\partial_+ A^+) + \Pi_u(\partial_+ u) + \Pi_\vartheta(\partial_+\vartheta) - \frac{1}{2}(\Pi^-)^2 \\ &\quad - \Pi^-(\partial_- A^-) - gA^-(\partial_-\phi) + (\partial_-\phi + gA^+)(\partial_+\phi) + \frac{1}{2}b^2 \\ &\quad + b(\partial_+ A^- - gA^+ + \Pi) + (\partial_+\bar{c})(\partial_+ c)\end{aligned} \tag{3.3}$$

Proceeding classically, the Euler–Lagrange equation for b reads

$$-b = (\partial_+ A^- - gA^+ + \Pi) \tag{3.4}$$

The requirement $\hat{\delta}b = 0$ [cf. (3.1c)] then implies

$$-\hat{\delta}b = \hat{\delta}(\partial_+ A^-) - g\hat{\delta}A^+ + \hat{\delta}\Pi = 0 \tag{3.5}$$

which in turn implies

$$\partial_+(\partial_+ c) = 0 \tag{3.6}$$

The above equation is also an Euler–Langrange equation obtained by the variation of $\mathscr{L}_{BRST}$ with respect to $\bar{c}$. We now define the bosonic momenta in the usual way so that[1]

$$\Pi^+ := \frac{\partial\mathscr{L}_{BRST}}{\partial(\partial_+ A^-)} = +b \tag{3.7}$$

The fermionic momenta are, however, defined using the directional derivatives such that[1]

$$\Pi_c := \mathcal{L}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\delta(\partial_+ c)} = \partial_+ \bar{c}; \qquad \Pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\delta(\partial_+ \bar{c})} \mathcal{L}_{\text{BRST}} = \partial_+ c \tag{3.8}$$

implying that the variable canonically conjugate to c is $(\partial_+ \bar{c})$ and the variable conjugate to $\bar{c}$ is $(\partial_+ c)$. In constructing the Hamiltonian density $\mathcal{H}_{\text{BRST}}$ from the Langrangian density in the usual way, one has to keep in mind that the former has to be Hermitian. Accordingly, we have[1]

$$\begin{aligned}\mathcal{H}_{\text{BRST}} &= \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi(\partial_+ \phi) + \Pi_u(\partial_+ u) + \Pi_\vartheta(\partial_+ \vartheta) \\ &\quad + \Pi_c(\partial_+ c) + (\partial_+ \bar{c})\, \Pi_{\bar{c}} - \mathcal{L}_{\text{BRST}} \\ &= \frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + gA^-(\partial_- \phi) - \frac{1}{2}(\Pi^+)^2 - \Pi^+(\Pi - gA^+) \\ &\quad + \Pi_c \Pi_{\bar{c}}\end{aligned} \tag{3.9}$$

We can check the consistency of (3.8) with (3.9) by looking at Hamilton's equations for the fermionic variables, i.e.,[1]

$$\partial_+ c = \frac{\overrightarrow{\partial}}{\partial \Pi_c} \mathcal{H}_{\text{BRST}}; \qquad \partial_+ \bar{c} = \mathcal{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} \tag{3.10}$$

Thus we see that

$$\partial_+ c = \frac{\overrightarrow{\partial}}{\partial \Pi_c} \mathcal{H}_{\text{BRST}} = \Pi_{\bar{c}}; \qquad \partial_+ \bar{c} = \mathcal{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} = \Pi_c \tag{3.11}$$

is in agreement with (3.8). For the operators c, $\bar{c}$, $\partial_+ c$, and $\partial_+ \bar{c}$, one needs to specify the anticommutation relations of $\partial_+ c$ with $\bar{c}$ or of $\partial_+ \bar{c}$ with c, but not of c with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that[1]

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0; \qquad \partial_+\{\bar{c}, c\} = 0 \tag{3.12a}$$

$$\{\partial_+ \bar{c}, c\} = -\{\partial_+ c, \bar{c}\} \tag{3.12b}$$

where {.,.} means an anticommulator. We thus see that the anticommutators in (3.12b) are nontrivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation[1]

$$[c, \mathcal{H}_{\text{BRST}}] = i\partial_+ c \tag{3.13}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathcal{H}_{\mathrm{BRST}}] = \{\partial_+\bar{c}\, c\}\partial_+ c \tag{3.14}$$

Equations (3.12)–(3.14) then imply

$$\{\partial_+\bar{c}, c\} = -\{\partial_+ c, \bar{c}\} = i \tag{3.15}$$

The minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory.[1,3,4]

3.3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformation (3.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\phi, Q] = \partial_+ c, \qquad [A^+, Q] = \partial_- c, \qquad [A^-, Q] = \partial_+ c \tag{3.16a}$$

$$[\Pi, Q] = g\partial_- c - \partial_-\partial_+ c, \qquad [\Pi^-, Q] = g\partial_+ c \tag{3.16b}$$

$$\{\bar{c}, Q\} = -(\Pi^+ + \Pi - \partial_-\phi - gA^+) \tag{3.16c}$$

$$\{\partial_+\bar{c}, Q\} = -(\partial_-\Pi^- - g\partial_-\phi) \tag{3.16d}$$

All other commutators and anticommutators involving Q vanish. In view of (3.16), the BRST charge operator for the present theory can be written as

$$Q = \int dx^- \, [ic(\partial_-\Pi^- - g\partial_-\phi) - i\partial_+ c\,(\Pi^+ + \Pi - \partial_-\phi - gA^+)] \tag{3.17}$$

This equation implies that the set of states satisfying the condition

$$\Pi^+|\psi> = 0 \tag{3.18a}$$

$$(\Pi - \partial_-\phi - gA^+)|\psi> = 0 \tag{3.18b}$$

$$(\partial_-\Pi^- - g\partial_-\phi)|\psi> = 0 \tag{3.18c}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belogs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory, we write the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider (3.6) (namely, $\partial_+\partial_+ c = 0$). The solution of this equation gives the Heisenberg operator $c(\tau)$ [and correspondingly $\bar{c}(\tau)$] as

$$c(\tau) = G\tau + F; \qquad \bar{c}(\tau) = G^\dagger\tau + F^\dagger \tag{3.19}$$

which at the light-cone time $\tau = 0$ implies

$$c \equiv c(0) = F, \qquad \bar{c} \equiv \bar{c}(0) = F^\dagger \tag{3.20a}$$

$$\partial_+ c \equiv \partial_+ c(0) = G; \qquad \partial_+\bar{c} \equiv \partial_+\bar{c}(0) = G^\dagger \tag{3.20b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_+\bar{c}, \partial_+ c\} = 0 \tag{3.21a}$$

$$\{\partial_+\bar{c}, c\} = i = -\{\partial_+ c, \bar{c}\} \tag{3.21b}$$

one then obtains

$$F^2 = F^{\dagger 2} = \{F^\dagger, F\} = \{G^+, G\} = 0 \tag{3.22}$$

$$\{G^\dagger, F\} = -\{G, F^\dagger\} = i \tag{3.23}$$

We now let $|0>$ denote the fermionic vacuum for which

$$G|0> = F|0> = 0 \tag{3.24}$$

Defining $|0>$ to have norm one, (3.23) implies

$$\langle 0|FG^\dagger|0\rangle = i; \qquad \langle 0|GF^\dagger|0\rangle = -i \tag{3.25}$$

so that

$$G^\dagger\,|0\rangle \neq 0; \qquad F^\dagger|0\rangle \neq 0 \tag{3.26}$$

The theory is thus seen to posses negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{\rm BRST}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators the Hamiltonian density is

$$\mathcal{H}_{\rm BRST} = \frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + gA^-(\partial_-\phi) - \frac{1}{2}(\Pi^+)^2$$
$$-\Pi^+(\Pi - gA^+) + G^\dagger G \tag{3.27}$$

and the BRST charge operator Q is

$$Q = \int dx^- \,[iF(\partial_-\Pi^- - g\partial_-\phi) - iG(\Pi^+ + \Pi - \partial_-\phi - gA^+)] \tag{3.28}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (3.18) holds, but also additional states for which

$$G|\psi\rangle = F|\psi\rangle = 0 \tag{3.29a}$$

$$\Pi^+|\psi\rangle \neq 0 \tag{3.29b}$$

$$(\Pi - \partial_-\phi - gA^+)|\psi\rangle \neq 0 \tag{3.29c}$$

$$(\partial_-\Pi^- - g\partial_-\phi)|\psi\rangle \neq 0 \tag{3.29d}$$

The Hamiltonian is, however, also invariant under the anti-BRST transformations (in which the role of c and $-\bar{c}$ gets interchanged) given by

$$\begin{aligned} &\bar{\hat{\delta}}\phi = 0; \quad \bar{\hat{\delta}}A^+ = -\partial_-\bar{c}; \quad \bar{\hat{\delta}}A^- = -\partial_+\bar{c} \\ &\bar{\hat{\delta}}u = -\partial_+\partial_+\bar{c}, \quad \bar{\hat{\delta}}\vartheta = 0 \end{aligned} \tag{3.30a}$$

$$\bar{\hat{\delta}}\Pi = -g(\partial_-\bar{c}); \quad \bar{\hat{\delta}}\Pi^+ = 0; \quad \bar{\hat{\delta}}\Pi^- = 0; \quad \bar{\hat{\delta}}\Pi_u = 0; \quad \bar{\hat{\delta}}\Pi_\vartheta = 0 \tag{3.30b}$$

$$\bar{\hat{\delta}}\bar{c} = 0; \quad \bar{\hat{\delta}}c = -b; \quad \bar{\hat{\delta}}b = 0 \tag{3.30c}$$

with generator or anti-BRST charge

$$\bar{Q} = \int dx^-[-i\bar{c}(\partial_-\Pi^- - g\partial_-\phi) + i\partial_+\bar{c}(\Pi^+ + \Pi - \partial_-\phi - gA^+)] \tag{3.31a}$$

$$= \int dx^-[-iF^\dagger(\partial_-\Pi^- - g\partial_-\phi) + iG^\dagger(\Pi^+ + \Pi - \partial_-\phi - gA^+)] \tag{3.31b}$$

We also have

$$[Q, H_{\text{BRST}}] = [\bar{Q}, H_{\text{BRST}}] = 0 \tag{3.32a}$$

$$H_{\text{BRST}} = \int dx^- \, \mathcal{H}_{\text{BRST}} \tag{3.32b}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \tag{3.33a}$$

$$\bar{Q}|\psi\rangle = 0 \tag{3.33b}$$

The states for which (3.18) hold satisfy both conditions (3.33a) and (3.33b) and in fact are the only states satisfying both of these conditions since, although with (3.22) and (3.23)

$$G^\dagger G = -GG^\dagger \tag{3.34}$$

there are no states of this operator with $G^\dagger|0\rangle = 0$ and $F^\dagger|0\rangle = 0$ [cf. (3.26)], and hence no free eigenstates of the fermionic part of $\mathcal{H}_{\mathrm{BRST}}$ which are annihilated by each of G, $G^\dagger$, F, $F^\dagger$. Thus the only states satisfying (3.33) are those satisfying the constraints (2.10) and (2.15).

Further, the states for which (3.18) holds satisfy both conditions (3.33a) and 3.33b) and in fact are the only states satisfying both of these conditions (3.33a) and (3.33b), because in view of (3.21), one cannot have simultaneously c, $\partial_+ c$, and $\bar{c}$, $\partial_+\bar{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (3.33) are those that satisfy the constraints of the theory (2.10) and (2.15), and they belong to the set of BRST-invariant and anti-BRST-invariant states. One can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (3.18) holds, but also additional states for which (3.29) holds. However, $\bar{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which (3.18) holds, simply because $G^\dagger|\psi\rangle \neq 0$ and $F^\dagger|\psi\rangle \neq 0$. Thus, in this alternative way also we see that the states satisfying $Q|\psi\rangle = \bar{Q}|\psi\rangle = 0$ [i.e., satisfying (3.33)] are only those that satisfy the constraints of the theory (2.10) and (2.15) and also that these states belong to the set of BRST invariant and anti-BRST-invariant states.

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APPENDIX

In this appendix, we describe briefly the Dirac quantization of the Schwinger model in the instant form of dynamics[9,1] (as studied in our previous work in ref. 1) under the gauge-fixing conditions $A_0 = 0$ and $A_1 = 0$. We show that the nonvanishing equal-time commutators of the theory in the gauge $A_0 = 0$ and $A_1 = 0$ are identical with the ones obtained by the Dirac quantization of the model under the gauge-fixing conditions $A_0 = 0$ and $A_1' = 0$, as they should, because the above two sets of gauge-fixing conditions conceptually mean the same.[10] The primes here denote space derivatives.

In the notations of ref. 1, the set of constraints of the theory under the gauge-fixing conditions $A_0 = 0$ and $A_1 = 0$ (see Section 2 of ref. 1) reads

$$\zeta_1 = \Omega_1 = \Pi_0 \approx 0 \tag{A.1a}$$

$$\zeta_2 = \Omega_2 = (E' + g\phi') \approx 0 \tag{A.1b}$$

$$\zeta_3 = A_0 \approx 0 \tag{A.1c}$$

$$\zeta_4 = A_1 \approx 0 \tag{A.1d}$$

By calculating the Poisson brackets among the constraints ζ_i one obtains the matrix

$$T_{\alpha\beta}(z, z') = \{\zeta_\alpha(z), \zeta_\beta(z')\}_P$$

$$= \begin{bmatrix} 0 & 0 & -\delta(z-z') & 0 \\ 0 & 0 & 0 & -\delta'(z-z') \\ \delta(z-z') & 0 & 0 & 0 \\ 0 & -\delta'(z-z') & 0 & 0 \end{bmatrix} \tag{A.2}$$

with the inverse

$$T^{-1}_{\alpha\beta}(z,z')$$

$$= \begin{bmatrix} 0 & 0 & \delta(z-z') & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\in(z-z') \\ -\delta(z-z') & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\in(z-z') & 0 & 0 \end{bmatrix} \tag{A.3}$$

with

$$\int dz\, T(x, z)\, T^{-1}(z, y) = 1_{4\times 4}\, \delta(x - y) \tag{A.4}$$

Finally, the nonvanishing equal-time commutators of the theory in the gauge $A_0 = 0$ and $A_1 = 0$ are obtained as

$$2[\phi(x), \Pi(y)] = \frac{2}{g}[E(x), \Pi(y)] = [A_1(x), E(y)] = 2i\delta(x - y) \tag{A.5}$$

These results are thus seen to be identical with the ones obtained by the Dirac quantization of the theory under the gauge-fixing conditions $A_0 = 0$ and $A_1' = 0$ and expressed by equation (2.23) of ref. 1. This implies clearly that the two sets of gauge-fixing conditions $A_0 = 0$ and $A_1 = 0$ and $A_0 = 0$

and $A_1^i = 0$ conceptually mean the same, as they should, at least in the context of a two-dimensional field theory like the present Schwinger model.[10]

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